Lecture 3 01/24/2018

Review of Electrostatics (Cont'd)

System of Conductors and Capacitance

Consider a system of N conductors with fixed geometry carrying charges $Q_1, \ldots, Q_N$ respectively. Their potential must have a linear dependence on their charges, i.e.,:

$$V_i = \sum_j P_{ij} Q_j$$

Here the coefficients P_{ij} are independent from V's and Q's; they only depend on the geometry and relative positions of the conductors. Note that the linearity in the relations between V's and Q's arises from the expressions for Φ in term of the Green's function. The matrix P, forming from P_{ij}, is invertible thereby resulting in:

$$Q_i = \sum_j C_{ij} V_j$$

The coefficients C_{ij} are known as coefficients of inductance, and the diagonal elements C_{ii}, are called capacities. They represent the capacitance of the individual conductors when all other conductors are grounded. For a pair of conductors, the capacitance is defined as follows:

$$C = \frac{Q}{\Delta V}$$

Here the charges on the two conductors are $\pm Q$, and ΔV is their potential difference.

The total potential energy in the electric field in the space outside the conductors follows:

$$W = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_i \sum_j C_{ij} V_i V_j = \frac{1}{2}\sum_i \sum_j P_{ij} Q_i Q_j$$

As a special case, let us consider a single conductor at potential V. In this case, we simply have $W = \frac{1}{2} C V^2$.

Next, consider a pair of conductors 1, 2 with charges $\pm Q$. Then:

$$W = \frac{1}{2}(P_{11}+P_{22})Q^2 + P_{12}(-Q^2) = \frac{1}{2}(P_{11}+P_{22}-2P_{12})Q^2$$

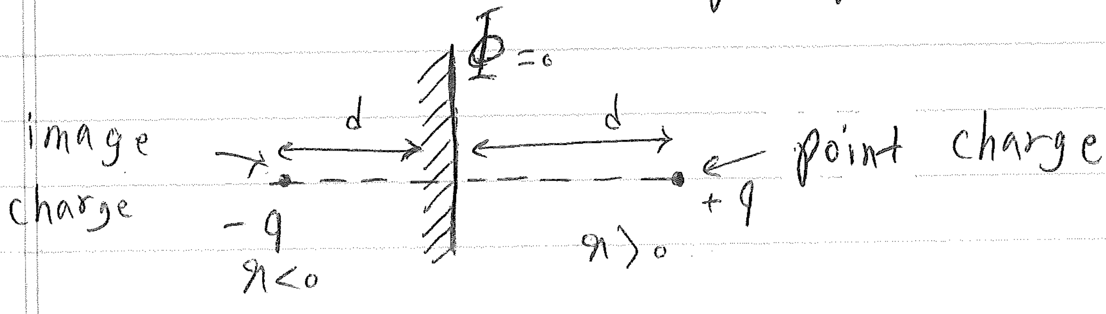
It can be shown that this is exactly equal to $\frac{1}{2}\frac{Q^2}{C}$, where C is the capacitance of the pair.

In general, we note that $C_{ij} = C_{ji}$ (due to the reciprocity theorem). Also, $C_{ii} > 0$ while $C_{ij} < 0$ for $i \neq j$.

Method of Images

The method of images provides a powerful approach for finding the potential for simple geometries that possess some symmetry. This is particularly the case for the Dirichlet problem for planar, cylindrical, and spherical geometries. Below, we consider these cases separately.

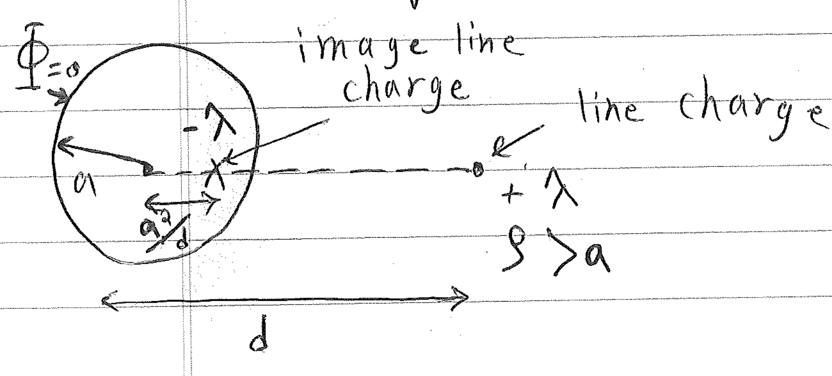
(1) Plane surface. This is a half-space problem, $x \geq 0$:



For a charge $+q$ in the region $x>0$, with $\Phi=0$ at $x=0$, we can use a point charge $-q$ in the region $x<0$ located at the image of the given charge. This gives rise to the same boundary condition, and hence the same potential everywhere at $x>0$ by the uniqueness of solution for a Dirichlet problem.

It is important to note that Φ in the region $x<0$ will be different in the presence of the image charge from that in the actual system. However, as long as we are interested in Φ in the $x>0$ region, the two setups give exactly the same answer.

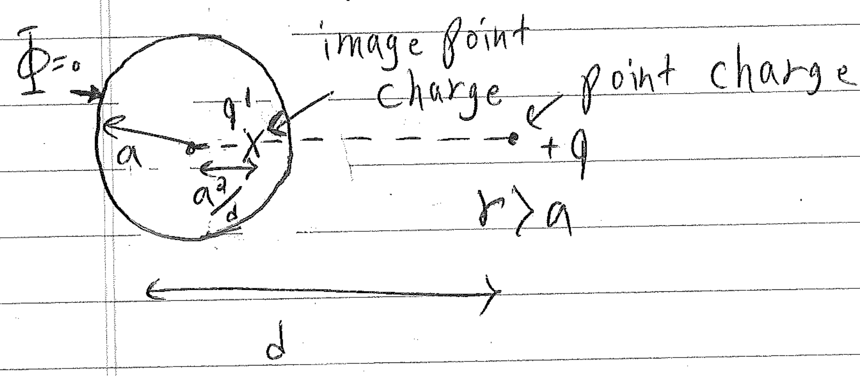
(2) Cylindrical surface and a line charge. The cross-sectional geometry of the problem is shown below:



λ: $\frac{\text{charge}}{\text{length}}$

The image line charge gives rise to the same boundary condition, $\Phi=0$ on the surface of the cylinder and exactly the same potential outside the cylinder ($\rho>a$).

(3) Spherical surface and a point charge. The cross-sectional geometry of this problem is as below:



$$\boxed{q' = -\frac{a}{d}q}$$

The image charge in this case is a point charge $q' = -\frac{a}{d}q$ situated at a distance $\frac{a^2}{d}$ from the center of the sphere. Again, the sum of the potential from the two charges gives the correct boundary condition, as well as Φ outside the sphere. One may consider simple variants of this problem. For example, if $\Phi=\Phi_0$ instead of 0, then we can add a third point charge $q'' = 4\pi\varepsilon_0 a\Phi_0$. The three charges together satisfy the required boundary condition, and hence give the correct Φ in the region $r \geq a$. One may also consider the reciprocal problem when a point charge $+q$ is situated at a distance $d < a$ inside a grounded sphere. In this case, the image charge is $q' = -\frac{a}{d} q$, which is located at a distance $d' = \frac{a^2}{d}$ from the center of the surface. The two charges together give the correct potential anywhere <u>inside</u> the sphere in this case.